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EXACT CONTROLLABILITY OF LINEAR DYNAMICAL SYSTEMS: A GEOMETRICAL APPROACH

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Abstract. In recent years there has been growing interest in the descriptive analysis of complex systems, permeating many aspects of daily life, obtaining considerable advances in the description of their structural and dynamical properties. However, much less effort has been devoted to studying the controllability of the dynamics taking place on them. Concretely, for complex systems it is of interest to study the exact controllability; this measure is defined as the minimum set of controls that are needed in order to steer the whole system toward any desired state. In this paper, we focus the study on the obtention of the set of all B making the system (A, B) exact controllable.

Keywords: controllability; exact controllability; eigenvalue; eigenvector; linear system

MSC 2010: 93B05, 93B27, 93B60

1. Introduction

In these recent years, the study of the control of complex networks with linear dynamics has gained importance in both science and engineering. Controllability of a dynamical system has been largely studied by several authors and under many different points of view, see [1], [2], [3], [5], [6], [4], [9] for example. Among different aspects in which we can study the controllability we have the notion of structural controllability that has been proposed by Lin [7] as a framework for studying the controllability properties of directed complex networks where the dynamics of the system is governed by a linear system: $\dot{x}(t) = Ax(t) + Bu(t)$; usually the matrix A of the system is linked to the adjacency matrix of the network, $x(t)$ is a time dependent vector of the state variables of the nodes, $u(t)$ is the vector of input signals, and B defines how the input signals are connected to the nodes of the network and is called the input matrix. Structurally controllable means that there exists a matrix $\bar{A}$ which is not allowed to contain a nonzero entry when the corresponding entry in A is zero

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such that the network can be driven from any initial state to any final state by appropriately choosing the input signals $u(t)$. Recent studies over the structural controllability can be found in [8].

In this paper, we analyze the exact controllability concept that, following [11], [10], is based on the maximum multiplicity, to identify the minimum set of driver nodes required to achieve full control of networks with arbitrary structures and link-weight distributions; we focus the study on the obtention of the set of all matrices B making the system $\dot{x}(t) = Ax(t) + Bu(t)$ exactly controllable. We have included several examples in order to make the work easier readable and it is completed with an example in the case of an undirected network.

2. Exact controllability

It is well known that many complex networks have linear dynamics and have a state space representation for its description:

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{2.1}$$

For simplicity, from now on we will write the system (2.1) as the pair of matrices (A, B).

There are many possible control matrices B in the system (2.1) that satisfy the controllability condition. The goal is to find the set of all possible matrices B, having the minimum number of columns corresponding to the minimum number $n_D(A)$ of independent controllers required to control the whole network.

Definition 1. Let A be a matrix. The exact controllability $n_D(A)$ is the minimum of the rank of all possible matrices B making the system 2.1 controllable:

$$n_D(A) = \min\{\operatorname{rank} B\colon\ \forall\, B \in M_{n\times i},\ 1 \leqslant i \leqslant n,\ (A, B) \text{ controllable}\}.$$

If no confusion is possible we will write simply n_D.

It is straightforward that n_D is invariant under similarity, that is to say: for any invertible matrix S we have $n_D(A) = n_D(S^{-1}AS)$. As a consequence, if necessary, we can consider A in its canonical Jordan form.

Example 1. 1) If $A = 0$, $n_D = n$.

2) If $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \neq \lambda_j$ for all $i \neq j$, then $n_D = 1$ (it suffices to take $B = (1 \ldots 1)^t$).

3) Not every matrix B having n_D columns makes the system controllable. For example if $A = \operatorname{diag}(1, 2, 3)$ and $B = (1, 0, 0)^t$, the system (A, B) is not controllable: $\operatorname{rank}\begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$, or equivalently $\operatorname{rank}\begin{pmatrix} A - \lambda I & B \end{pmatrix} = 2$ for $\lambda = 2, 3$.

Proposition 1 ([11]). *We have*

$$n_D = \max_i \{\mu(\lambda_i)\},$$

where $\mu(\lambda_i) = \dim \operatorname{Ker}(A - \lambda_i I)$ *is the geometric multiplicity of the eigenvalue* λ_i*.*

3. Constructing the controllability space

Given a matrix A, we will try to get all matrices B with the smallest possible size, making the system (A, B) controllable. This study is of interest, because as we saw in 1)–3) not every matrix B is useful for the system being controllable.

Following Proposition 1, the problem is linked to the eigenstructure of the matrix A.

First of all we want to note that given a vector subspace F of a vector space E, if we consider two projections P_i, $i = 1, 2$, onto any two complementary subspaces G_i, $i = 1, 2$, along the subspace F we have that for all $v \in E$, $P_1(v) \neq 0$ if and only if $P_2(v) \neq 0$. So, in the case where the required information is only whether a vector is in F or not, we can define the projection P onto $E \setminus F$ along F as the projection over any complementary subspace G of F along F.

Proposition 2. *Let* A *be a matrix having* r *eigenvalues* $\lambda_1, \ldots, \lambda_r$ *with geometric multiplicity one for each of them, and with algebraic multiplicities* $n_1, \ldots, n_r$*. Then* $n_D = 1$*. Moreover, for* $i = 1, \ldots, r$ *let* P_i *be the projection onto* $\operatorname{Ker}(A - \lambda_i I)^{n_i} \setminus \operatorname{Ker}(A - \lambda_i I)^{n_i - 1}$ *along* $\bigoplus_{j \neq i} \operatorname{Ker}(A - \lambda_i I)^{n_j} \oplus \operatorname{Ker}(A - \lambda_i I)^{n_i - 1}$*. Then, for an* $n \times 1$ *matrix* B*, the pair* (A, B) *is controllable if and only if* $P_i B \neq 0$ *for every* $i = 1, \ldots, r$*.*

Proof. We consider the equivalent Jordan form

$$J = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 & & & & & & \\ 1 & \lambda_1 & \ldots & 0 & 0 & & & & & & \\ & \ddots & \ddots_{(n_1)} & & & & & & & & \\ 0 & 0 & \ldots & \lambda_1 & 0 & & & & & & \\ 0 & 0 & \ldots & 1 & \lambda_1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & \lambda_r & 0 & \ldots & 0 & 0 \\ & & & & & & 1 & \lambda_r & \ldots & 0 & 0 \\ & & & & & & & \ddots & \ddots_{(n_r)} & & \\ & & & & & & 0 & 0 & \ldots & \lambda_r & 0 \\ & & & & & & 0 & 0 & \ldots & 1 & \lambda_r \end{pmatrix},$$

and the associated Jordan basis constructed as follows:

$$\begin{aligned} v_{1_i} &\in \operatorname{Ker}(A-\lambda_i I)^{n_i} \setminus \operatorname{Ker}(A-\lambda_i I)^{n_i-1}, \\ v_{2_i} &= (A-\lambda_i I)v_{1_i}, \\ &\vdots \\ v_{n_i} &= (A-\lambda_i I)^{n_i-1}v_{1_i}, \end{aligned}$$

for each $i = 1, \dots, r$.

Clearly,

$$\operatorname{rank}(A-\lambda I) = \operatorname{rank}(J-\lambda I) = \begin{cases} n & \text{for } \lambda \neq \lambda_1, \dots, \lambda_r, \\ n-1 & \text{for } \lambda = \lambda_1, \dots, \lambda_r. \end{cases}$$

Then $n_D = 1$.

For any $u \in \mathbb{R}^n$ we consider $B = [u]$. Then $u = \sum_{ji} \alpha_{ji} v_{ji}$ and $P_i u = \alpha_{1i} v_{1i}$.

Finally, it is easy to compute

$$\operatorname{rank}(A-\lambda_i I \quad B) = n \quad \text{if and only if } P_i B \neq 0.$$

□

E x a m p l e 2. Let

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

be the matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$ and the respective multiplicities $n_1 = 3$ and $n_2 = 2$.

Let $v_{11} = (0,0,1,0,0) \in \operatorname{Ker}(A-2I)^3 \setminus \operatorname{Ker}(A-2I)^2$ and $v_{12} = (-139,-14,1,1,1) \in \operatorname{Ker}(A-2I)^2 \setminus \operatorname{Ker}(A-2I)$. The Jordan basis is $v_{11} = (0,0,1,0,0)$, $v_{21} = (4,3,0,0,0)$, $v_{31} = (9,0,0,0,0)$, $v_{12} = (-139,-14,1,1,1)$, $v_{22} = (112,26,6,2,0)$.

Then $\operatorname{Im} B = [u]$ with $u = \alpha_{11}v_{11} + \alpha_{21}v_{21} + \alpha_{31}v_{31} + \alpha_{12}v_{12} + \alpha_{22}v_{22}$ is such that

$$\begin{aligned} &\operatorname{rank}(A-\lambda I \;\; B) = 5 \quad \forall \lambda \neq 2,3, \\ &\operatorname{rank}(A-2I \;\; B) = 5 \quad \text{if and only if } \alpha_{11}v_{11} = P_1 B \neq 0, \\ &\operatorname{rank}(A-3I \;\; B) = 5 \quad \text{if and only if } \alpha_{12}v_{12} = P_2 B \neq 0. \end{aligned}$$

In the previous results it can be observed that we cannot control the system with a single control if the matrix A has more than one independent eigenvector corresponding to the same eigenvalue. Then we will try to write for this case all the matrices B which control the system. As we can see in the following example, the study is slightly more sensitive.

E x a m p l e 3. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

be the matrix with a unique eigenvalue $\lambda_0 = 0$. We have

$$\operatorname{rank}(A) = 5 < 8.$$

Then $n_D = 3$.

If we consider $u_1 = (1,1,1,1,1,1,1,1)$, $u_2 = (2,1,1,1,1,1,1,1) \in \operatorname{Ker}(A-\lambda_0 I)^3 \setminus \operatorname{Ker}(A-\lambda_0 I)^2$ and $u_3 = (0,1,2,0,1,1,1,1) \in \operatorname{Ker}(A-\lambda_0 I)^2 \setminus \operatorname{Ker}(A-\lambda_0 I)$, then

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \operatorname{rank}(B) = 3, \ \operatorname{rank}(A-\lambda_0 I \quad B) = 8.$$

But, if we consider $v_1 = (1,1,1,1,1,1,1,1)$, $v_2 = (1,2,2,1,2,2,1,2) \in \operatorname{Ker}(A-\lambda_0 I)^3 \setminus \operatorname{Ker}(A-\lambda_0 I)^2$ and $v_3 = (0,1,2,0,1,1,1,1) \in \operatorname{Ker}(A-\lambda_0 I)^2 \setminus \operatorname{Ker}(A-\lambda_0 I)$, then

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad \operatorname{rank}(B) = 3, \ \operatorname{rank}(A-\lambda_0 I \quad B) = 7 < 8.$$

Therefore, we should specify a little more how to determine the matrix B.

Proposition 3. *Let A be a matrix with a single eigenvalue λ_0 with geometric multiplicity δ and the orders of the Jordan blocks $k_1 \geqslant \ldots \geqslant k_\delta$. Then $n_D = \delta$. Moreover, for any $n \times \delta$ matrix $B = [u_1, \ldots, u_\delta]$, the pair (A, B) is controllable if and only if $P_j u_{i_j} \neq 0$ for all $j = 1, \ldots, \delta$ $((i_1, \ldots, i_\delta)$ being some possible required reordering of $(1, \ldots, \delta))$, where P_j is the projection onto $\operatorname{Ker}(A - \lambda_0 I)^{k_j} \setminus \operatorname{Ker}(A - \lambda_0 I)^{k_j - 1} \bigoplus\limits_{l=1,\ldots,j-1} [(A - \lambda_0 I)^{k_l - k_j} u_l]$ along $(A - \lambda_0 I)^{k_j - 1} \bigoplus\limits_{l=1,\ldots,j-1} [(A - \lambda_0 I)^{k_l - k_j} u_l]$.*

P r o o f. The matrix A in the basis

$$\begin{matrix} u_1 & (A - \lambda_0 I)u_1 & \ldots & (A - \lambda_0 I)^{k_1 - 1} u_1 \\ \vdots & & & \\ u_\delta & (A - \lambda I_0)u_\delta & \ldots & (A - \lambda_0 I)^{k_\delta - 1} u_\delta \end{matrix}$$

where u_i are chosen in such a way that the collection of the vectors are linearly independent, has the Jordan form

$$J(\lambda_0) = \begin{pmatrix} \lambda_0 & & & & & & \\ 1 & \ddots^{(k_1)} & & & & & \\ & 1 & \lambda_0 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_0 & & \\ & & & & 1 & \ddots^{(k_\delta)} & \\ & & & & & 1 & \lambda_0 \end{pmatrix}.$$

Clearly $\operatorname{rank}(J - \lambda_0 I) = n - \delta$. Then $n_D = \delta$.

If $P_j u_{i_j} \neq 0$ for all $j = 1, \ldots, \delta$, then

$$\begin{matrix} u_{i_1} & (A - \lambda_0 I)u_{i_1} & \ldots & (A - \lambda_0 I)^{k_1 - 1} u_{i_1} \\ \vdots & & & \\ u_{i_\delta} & (A - \lambda I_0)u_{i_\delta} & \ldots & (A - \lambda_0 I)^{k_\delta - 1} u_{i_\delta} \end{matrix}$$

is a Jordan basis and in this basis $\begin{pmatrix} A - \lambda_0 I & B \end{pmatrix}$ takes the form

$$\begin{pmatrix} \lambda_0 & & & & & & & 1 & 0 & \ldots & 0 \\ 1 & \ddots^{(k_1)} & & & & & & 0 & \vdots & & \vdots \\ & 1 & \lambda_0 & & & & & \vdots & 1 & & 0 \\ & & & \ddots & & & & \vdots & 0 & & 0 \\ & & & & \lambda_0 & & & 0 & 0 & & 1 \\ & & & & 1 & \ddots^{(k_\delta)} & & \vdots & \vdots & & \vdots \\ & & & & & 1 & \lambda_0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

whose rank is n and the pair (A,B) is controllable.

Conversely, let $B=\begin{pmatrix}\alpha_{11} & \dots & \alpha_{\delta_1}\\ \vdots & & \vdots\\ \alpha_{1n} & \ddots & \alpha_{\delta n}\end{pmatrix}$ be the matrix in the Jordan basis.

If

$$\operatorname{rank}(A-\lambda_0 I \quad B)=n,$$

then the minor Δ^1 is

$$\Delta^1=\begin{vmatrix}\alpha_{11} & \dots & \alpha_{\delta 1}\\ \alpha_{1k_1+1} & & \alpha_{\delta k_1+1}\\ \vdots & & \vdots\\ \alpha_{1\sum_1^{\delta-1}k_i+1} & & \alpha_{\delta\sum_1^{\delta-1}k_i+1}\end{vmatrix}\neq 0.$$

Let u_{i_1} be such that $\alpha_{i_1}\neq 0$ and $\Delta^2=\Delta^1_{i_1}\neq 0$. Then $P_1u_{i_1}\neq 0$.

Taking into account that $\Delta^2\neq 0$, there is $u_{j_2}\neq u_{i_1}$ such that $\alpha_{jk_1+1}\neq 0$ and $\Delta^3=\Delta^2_{jk_1+1}\neq 0$, so $P_2u_{j_2}\neq 0$. Following this process, we show the result. □

E x a m p l e 4. Following Example 3, we have that in the first case $u_2\in \operatorname{Ker}(A-\lambda_0 I)^3\setminus \operatorname{Ker}(A-\lambda_0 I)^2\oplus[u_1]$ and $P_2u_2\neq 0$; and $u_3=(0,1,2,0,1,1,1,1)\in\operatorname{Ker}(A-\lambda_0 I)^2\setminus\operatorname{Ker}(A-\lambda_0 I)\oplus[(A-\lambda_0)u_1]\oplus[(A-\lambda_0)u_2]$ and $P_3u_3\neq 0$.

Nevertheless, in the second case $v_2\in\operatorname{Ker}(A-\lambda_0 I)^2\oplus[v_1]$ and $P_2v_2=0$.

Finally, we analyze the general case, where the matrix A has multiple eigenvalues with multiple independent eigenvectors for some (or all) eigenvalues.

Proposition 4. *Let A be a matrix having r eigenvalues $\lambda_1,\dots,\lambda_r$ with algebraic multiplicities $n_1\geqslant\dots\geqslant n_r$, geometric multiplicities $\delta_1\geqslant\dots\geqslant\delta_r$, respectively, and order of Jordan blocks for each eigenvalue $k_{11}\geqslant\dots\geqslant k_{1\delta_1},\dots,k_{r1}\geqslant\dots\geqslant k_{r\delta_r}$. Then $n_D(A)=\delta_1$. Moreover, for any $n\times\delta_1$ matrix $B=[u_1,\dots u_{\delta_1}]$, the pair (A,B) is controllable if and only if $P_{lj}u_{ij}\neq 0$ for $j\leqslant\delta_l$ ($(i_1,\dots,i_{\delta_l})$ being some possible required reordering of $(1,\dots,\delta_l)$), where P_{lj} is the projection onto* $\operatorname{Ker}(A-\lambda_l I)^{k_{lj}}\setminus\operatorname{Ker}(A-\lambda_l I)^{k_{lj}-1}\bigoplus_{\nu_j=1,\dots,l_j-1}[(A-\lambda_l I)^{k_{\nu j}-k_{lj}}u_\nu]\bigoplus_{\mu\neq l}\operatorname{Ker}(A-\lambda_\mu I)^{n_\mu}$.

P r o o f. Writing the pair (A,B) in a Jordan basis, we have

$$(J,B)=\left(\begin{pmatrix}J_1 & & \\ & \ddots & \\ & & J_r\end{pmatrix},\begin{pmatrix}B_1\\ \vdots\\ B_r\end{pmatrix}\right),$$

where $J_i(\lambda_i)$ is as $J(\lambda_0)$ in Proposition 3 and B_i are blocks corresponding to the block sizes $J_i(\lambda_i)$ of J.

It is easy to observe that (J,B) is controllable if and only if $(J_i(\lambda_i),B_i)$ is controllable. Then it suffices to apply Proposition 3. □

Example 5. Let

$$A=\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3
\end{pmatrix}.$$

Taking $\mathrm{Im}B = [u_1, u_2]$ with $u_1 = (1,0,0,0,0,0,0,0,1,0,0,0,0,0,1)$ and $u_2 = (0,0,0,0,0,1,0,0,0,0,0,0,1,0,0)$, it is easy to observe that $\operatorname{rank}\begin{pmatrix} A-\lambda I & B\end{pmatrix} = 15$ for all λ.

4. Example of description of the set of drivers for an undirected network

We illustrate the work applying it to a simple example of an undirected graph represented in Figure 1.

The adjacency matrix of the graph is

$$A=\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1\\
1 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

whose eigenvalues are $\lambda_1 = -2.0861$, $\lambda_2 = -1.0000$, $\lambda_3 = 0.0000$, $\lambda_4 = 0.0000$, $\lambda_5 = 0.5720$, $\lambda_6 = 2.5141$, and $\dim \operatorname{Ker} A = 2$. Then $n_D = 2$.

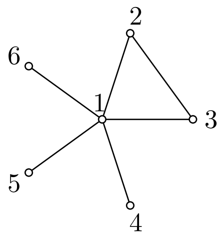

Figure 1. Example of an undirected graph.

The corresponding eigenvectors are

$$
\begin{aligned}
u_1 &= (0.7256, -0.2351, -0.2351, -0.3478, -0.3478, -0.3478),\\
u_2 &= (0, -0.7071, 0.7071, 0.0000, -0.0000, 0),\\
u_3 &= (-0.0000, 0.0000, 0.0000, -0.6643, -0.0790, 0.7433),\\
u_4 &= (0.0000, -0.0000, -0.0000, 0.4747, -0.8127, 0.3379),\\
u_5 &= (-0.2178, 0.5088, 0.5088, -0.3807, -0.3807, -0.3807),\\
u_6 &= (0.6527, 0.4311, 0.4311, 0.2596, 0.2596, 0.2596).
\end{aligned}
$$

The set of matrices B having minimal number of columns making the system (A, B) controllable is

$$B = \begin{pmatrix} \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_5 u_5 + \alpha_6 u_6 & \alpha_4 u_4 \end{pmatrix}$$

with $\alpha_i \neq 0$ for all $i = 1, \dots, 6$.

The controllability matrix $\mathcal{C}$ is

columns 1 and 2

$$
\left(\begin{array}{cc}
0.7256\alpha_1 - 0.2178\alpha_5 + 0.6527\alpha_6 & 0\\
0.5088\alpha_5 - 0.7071\alpha_2 - 0.2351\alpha_1 + 0.4311\alpha_6 & 0\\
0.7071\alpha_2 - 0.2351\alpha_1 + 0.5088\alpha_5 + 0.4311\alpha_6 & 0\\
0.2596\alpha_6 - 0.6643\alpha_3 - 0.3807\alpha_5 - 0.3478\alpha_1 & 0.4747\alpha_4\\
0.2596\alpha_6 - 0.079\alpha_3 - 0.3807\alpha_5 - 0.3478\alpha_1 & -0.8127\alpha_4\\
0.7433\alpha_3 - 0.3478\alpha_1 - 0.3807\alpha_5 + 0.2596\alpha_6 & 0.3379\alpha_4
\end{array}\right.
$$

columns 3 and 4

$$
\begin{array}{cc}
1.641\alpha_6 - 0.1245\alpha_5 - 1.5136\alpha_1 & -0.0001\alpha_4\\
0.4905\alpha_1 + 0.7071\alpha_2 + 0.291\alpha_5 + 1.0838\alpha_6 & 0\\
0.4905\alpha_1 - 0.7071\alpha_2 + 0.291\alpha_5 + 1.0838\alpha_6 & 0\\
0.7256\alpha_1 - 0.2178\alpha_5 + 0.6527\alpha_6 & 0\\
0.7256\alpha_1 - 0.2178\alpha_5 + 0.6527\alpha_6 & 0\\
0.7256\alpha_1 - 0.2178\alpha_5 + 0.6527\alpha_6 & 0
\end{array}
$$

columns 5 and 6

$$\begin{matrix}
3.1578\alpha_1 - 0.0714\alpha_5 + 4.1257\alpha_6 & 0\\
0.1665\alpha_5 - 0.7071\alpha_2 - 1.0231\alpha_1 + 2.7248\alpha_6 & -0.0001\alpha_4\\
0.7071\alpha_2 - 1.0231\alpha_1 + 0.1665\alpha_5 + 2.7248\alpha_6 & -0.0001\alpha_4\\
0.1641\alpha_6 - 0.1245\alpha_5 - 1.5136\alpha_1 & -0.0001\alpha_4\\
1.641\alpha_6 - 0.1245\alpha_5 - 1.5136\alpha_1 & -0.0001\alpha_4\\
1.641\alpha_6 - 0.1245\alpha_5 - 1.5136\alpha_1 & -0.0001\alpha_4
\end{matrix}$$

columns 7 and 8

$$\begin{matrix}
10.3726\alpha_6 - 0.0405\alpha_5 - 6.587\alpha_1 & -0.0005\alpha_4\\
2.1347\alpha_1 + 0.7071\alpha_2 + 0.0951\alpha_5 + 6.8505\alpha_6 & -0.0001\alpha_4\\
2.1347\alpha_1 - 0.7071\alpha_2 + 0.0951\alpha_5 + 6.8505\alpha_6 & -0.0001\alpha_4\\
3.1578\alpha_1 - 0.0714\alpha_5 + 4.1257\alpha_6 & 0\\
3.1578\alpha_1 - 0.0714\alpha_5 + 4.1257\alpha_6 & 0\\
3.1578\alpha_1 - 0.0714\alpha_5 + 4.1257\alpha_6 & 0
\end{matrix}$$

columns 9 and 10

$$\left.\begin{matrix}
13.7428\alpha_1 - 0.0240\alpha_5 + 26.0781\alpha_6 & -0.0002\alpha_4\\
0.0546\alpha_5 - 0.7071\alpha_2 - 4.4523\alpha_1 + 17.2231\alpha_6 & -0.0006\alpha_4\\
0.7071\alpha_2 - 4.4523\alpha_1 + 0.0546\alpha_5 + 17.2231\alpha_6 & -0.0006\alpha_4\\
10.3726\alpha_6 - 0.0405\alpha_5 - 6.587\alpha_1 & -0.0005\alpha_4\\
10.3726\alpha_6 - 0.0405\alpha_5 - 6.587\alpha_1 & -0.0005\alpha_4\\
10.3726\alpha_6 - 0.0405\alpha_5 - 6.587\alpha_1 & -0.0005\alpha_4
\end{matrix}\right)$$

with rank($\mathcal{C}$) if and only if $\alpha_i = 6$ for all $i = 1, \ldots, 6$.

In particular, for $\alpha_i = 1$ for all $i = 1, \ldots, 6$ the controllability matrix is

$$\mathcal{C} = \begin{pmatrix}
1.1605 & 0 & 0.0029 & -0.0001 & 7.2121 & 0 & 3.7451 & -0.0005 & 39.7969 & -0.0002\\
-0.0023 & 0 & 2.5724 & 0 & 1.1611 & -0.0001 & 9.7874 & -0.0001 & 12.1183 & -0.0006\\
1.4119 & 0 & 1.1582 & 0 & 2.5753 & -0.0001 & 8.3732 & -0.0001 & 13.5325 & -0.0006\\
-1.1332 & 0.4747 & 1.1605 & 0 & 0.0029 & -0.0001 & 7.2121 & 0 & 3.7451 & -0.0005\\
-0.5479 & -0.8127 & 1.1605 & 0 & 0.0029 & -0.0001 & 7.2121 & 0 & 3.7451 & -0.0005\\
0.2744 & 0.3379 & 1.1605 & 0 & 0.0029 & -0.0001 & 7.2121 & 0 & 3.7451 & -0.0005
\end{pmatrix}.$$

It is easy to observe that if some $\alpha_i = 0$ in the matrix $\mathcal{C}$, then the matrix does not have a full rank, as well as if we consider $\operatorname{Im} B = [\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_5 u_5 + \alpha_6 u_6 + \alpha_4 u_4]$, the system is not controllable.

5. Conclusion

In this work, given an n-order square matrix A, we have explicitly described a way how to obtain all possible matrices B having the minimum number of columns, making the system (A, B) controllable. Several examples have been included in order to make the work easier to read and it is completed with an example in the case of an undirected network.

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